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Distributive Extensions of Modules

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Let M and N modules over a commutative ring R, with M a submodule of N. We say $M \subseteq N$ is *distributive* if the following equivalent conditions are satisfied.

$$(X+Y)\cap M = (X\cap M) + (Y\cap M), \quad \text{for all submodules } X, Y \text{ of } N;$$

$$(X\cap Y) + M = (X+M)\cap(Y+M), \quad \text{for all submodules } X, Y \text{ of } N.$$

We shall also say that N is a "distributive extension" of M and that M is a "distributive submodule" of N.

The distributive submodules of a given module form a sublattice of the lattice of all submodules. In Section 1 we show that every distributive submodule of a finitely generated module M is stable under all endomorphisms of M.

It is shown in [2] that every R-module which is locally non-zero at every maximal ideal of R, has a maximal distributive extension and the question is raised: is this unique up to isomorphism; that is, is there a "distributive hull"? In Section 2 we show that over Noetherian arithmetic rings, this is indeed the case. Over a local Noetherian arithmetic ring, either the injective hull or the module itself is a distributive hull.

In Section 3 we consider distributive extensions of modules over an integral domain. It turns out that every module of rank greater than one has no proper distributive submodules, while, if $M \subseteq N$ is a distributive extension of rank one modules, then M and N have the same torsion submodule and coincide locally at the prime ideals belonging to its support.

In Section 4 we describe the distributive hull $D(M)$ in certain cases. In particular, one has the following for a module M over a Dedekind domain R (Dedekind domains are precisely the Noetherian arithmetic domains). If M has rank zero, then $D(M) = (\coprod_{P\in C} K/R_P) \amalg (\coprod_{P\notin C} M_P)$, where K is the field of fractions of R and C is the set of maximal ideals P of R for which the R_P-module M_P is cyclic. (For $P \in C$, K/R_P is the injective hull of M_P.) If M has rank one and the ideal class group of R is a torsion group, then

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$D(M) = S_M^{-1}M$, where S_M is the (multiplicatively closed) set of elements of R which are not zero-divisors on M. In fact "$D(M) = S_M^{-1}M$ for all rank one M" is a necessary and sufficient condition for the ideal class group of R to be a torsion group. If M has rank greater than one, then, as already noted, $D(M) = M$.

Throughout, the term "ring" will mean "commutative ring with identity," and we shall use the following notation: if R is a ring, Spec(R) is the set of all prime ideals of R; if M is an R-module, $\operatorname{Supp}(M) = \{P \in \operatorname{Spec}(R) \mid M_P \neq 0\}$, $\operatorname{Ann}(M) = \{r \in R \mid rm = 0, \text{ all } m \in M\}$; if X is a submodule of M and $x, y \in M$, $(X : y) = \{r \in R \mid ry \in X\}$, $(x : y) = (Rx : y)$, $\operatorname{Ann}(y) = (0 : y)$.

1

PROPOSITION 1.1. *Let $M \subseteq N$ be an extension of R-modules. Then $M \subseteq N$ is distributive if, and only if,*

$$(M : y) + (y : x) = R, \qquad \textit{for all } x \in M \textit{ and all } y \in N.$$

Proof. Let $x \in M$, $y \in N$. Then for any $r \in R$,

$$r \in (M : y) \Leftrightarrow r(x - y) \in M \cap R(x - y);$$

$$1 - r \in (y : x) \Leftrightarrow x = r(x - y) + sy, \qquad \text{for some } s \in R.$$

Therefore

$$(M : y) + (y : x) = R \Leftrightarrow \text{there exist } r, s \in R \text{ such that } r(x - y) \in M \cap R(x - y) \text{ and } x = r(x - y) + sy.$$

Suppose $M \subseteq N$ is distributive and let $x \in M$, $y \in N$. Then

$$x = (x - y) + y \in M \cap (R(x - y) + Ry) = M \cap R(x - y) + M \cap Ry.$$

Therefore $(M : y) + (y : x) = R$, by the above.

Conversely, suppose $(M : y) + (y : x) = R$ for all $x \in M$ and all $y \in N$. Let X, Y be submodules of N. Let $x \in (X + Y) \cap M$. Then $x \in M$ and $x = z + y$ with $z \in X$, $y \in Y$. By the above, $x = rz + sy$ for some r, $s \in R$ with $rz \in M$ and hence $sy \in M$. Thus $x \in (Rz \cap M) + (Ry \cap M) \subseteq (X \cap M) + (Y \cap M)$. Therefore $(X + Y) \cap M = (X \cap M) + (Y \cap M)$.

COROLLARY (cf. [2, Lemma 2.7]). *Let R be a local ring. Then $M \subseteq N$ is distributive if, and only if, $M \subseteq Ry$ for all $y \in N - M$ (equivalently, if, and only if, M is comparable by inclusion with every submodule of N).*

In particular, a module over a field has no proper distributive submodules.

PROPOSITION 1.2. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{M}$, and let M be an R-module. Then every proper distributive submodule of M is of the form $\mathfrak{M}y$, for some $y \in M$.*

Proof. Let X be a proper distributive submodule of M. Then taking an element $z \in M - X$, we have $X \subseteq Rz$, by the corollary to Proposition 1.1. But R is Noetherian. Therefore X is finitely generated, and so we may assume that M is finitely generated. Then $X = \mathfrak{M}^k M$, where k is a positive integer ([1, Proposition 9]). Let $Y = \mathfrak{M}^{k-1}M$. Then X is a distributive submodule of Y and $X \neq Y$, by Nakayama's lemma. Thus we can choose an element $y \in Y - X$ and then, by the corollary to Proposition 1.1, $X \subseteq \mathfrak{M}y \subseteq \mathfrak{M}Y = X$.

Let M be a finitely generated R-module and let f be an R-endomorphism of M. It is known that if every submodule of M is a distributive submodule, then every submodule of M is stable under f ([6, Proposition 4.3]). The next result shows that this implication is true on the submodules individually.

PROPOSITION 1.3. *Let M be a module over a ring R. If X is a distributive submodule of M, then X is stable under every R-endomorphism of M which satisfies a monic polynomial with coefficients in R.*

Proof. Let f be an R-endomorphism of M and suppose that $f^n = \sum_{i=0}^{n-1} a_i f^i$, for some positive integer n and elements $a_0, a_1, \ldots, a_{n-1} \in R$.
Let $x \in X$. Then by Proposition 1.1, $(X : f(x)) + (f(x) : x) = R$. Therefore there exist elements $r, s, t \in R$ with $1 = s + t$, $sx = rf(x)$, $tf(x) \in X$. Since f is an R-module morphism, we have by induction that $s^i x = r^i f^i(x)$, for all positive integers i. Therefore $r^{n-1} f^i(x) \in Rx$, for $i = 0, 1, \ldots, n-1$. Therefore $r^{n-1} f^n(x) \in Rx$, by hypothesis, and hence $s^{n-1} f(x) = f(s^{n-1}x) = f(r^{n-1} f^{n-1}(x)) = r^{n-1} f^n(x) \in Rx \subseteq X$. Therefore $f(x) = (s + t)^{n-1} f(x) \in X$. Hence $f(X) \subseteq X$.

COROLLARY. *Let M be a finitely generated R-module and let X be a distributive submodule of M. Then X is stable under every R-endomorphism of N.*

2

We turn now to the question of the uniqueness of a maximal distributive extension of a given R-module M. We shall see that M has no maximal distributive extensions at all unless $M_{\mathfrak{M}} \neq 0$ for all maximal ideals $\mathfrak{M}$ of R. So in order to include all modules in the discussion, we make the following definition. An extension $M \subseteq N$ of R-modules will be called *supporting* if $M_{\mathfrak{M}} \neq 0$ for all maximal ideals $\mathfrak{M}$ of R for which $N_{\mathfrak{M}} \neq 0$. Thus all extensions of M are supporting if, and only if, $M_{\mathfrak{M}} \neq 0$ for all maximal ideals $\mathfrak{M}$ of R (for if $M_{\mathfrak{M}} = 0$, then $M \subseteq M \oplus R/\mathfrak{M}$ is not supporting).

Using elementary properties of modules of fractions, the following is an immediate consequence of the definition of "distributive extension."

LEMMA 2.1 (cf. [2, Lemmas 2.5 and 2.6]). *Let $M \subseteq N$ be an extension of R-modules.*

(i) *Let S be a multiplicatively closed subset of R. If $M \subseteq N$ is distributive as R-modules, then $S^{-1}M \subseteq S^{-1}N$ is distributive as $S^{-1}R$-modules.*

(ii) *$M \subseteq N$ is distributive as R-modules if, and only if, $M_P \subseteq N_P$ is distributive as R_P-modules for all prime/maximal ideals P of R.*

PROPOSITION 2.2. *If an extension of R-modules $M \subseteq N$ is distributive and supporting then it is essential.*

Proof. Suppose $M \subseteq N$ is distributive and supporting. Let X be a non-zero submodule of N. Then $X_{\mathfrak{M}} \neq 0$ for some maximal ideal $\mathfrak{M}$ of R. Then $N_{\mathfrak{M}} \neq 0$ and so $M_{\mathfrak{M}} \neq 0$, as $M \subseteq N$ is supporting. But by Lemma 2.1, $M_{\mathfrak{M}} \subseteq N_{\mathfrak{M}}$ is a distributive extension of modules over the local ring $R_{\mathfrak{M}}$. Therefore $M_{\mathfrak{M}} \subseteq X_{\mathfrak{M}}$ or $X_{\mathfrak{M}} \subseteq M_{\mathfrak{M}}$, by the corollary to Proposition 1.1. Hence $(M \cap X)_{\mathfrak{M}} = M_{\mathfrak{M}} \cap X_{\mathfrak{M}} \neq 0$. Therefore $M \cap X \neq 0$.

COROLLARY 1. *Let M be an R-module. Then every distributive and supporting extension of M is contained in a maximal distributive and supporting extension of M. More precisely, if M is a distributive and supporting submodule of an R-module N, then N is a submodule of an R-module X which has the properties* (a) *$M \subseteq X$ is distributive and supporting, and* (b) *$X \subseteq Y$, with $M \subseteq Y$ distributive and supporting, implies $X = Y$.*

In particular, every module has maximal distributive and supporting extensions.

Proof. By the proposition, every distributive and supporting extension of M can be embedded in the injective hull $E(M)$ of M. Suppose then that $M \subseteq N \subseteq E(M)$ and let W denote the set of submodules of $E(M)$ which contain N and are distributive and supporting extensions of M. Because every distributive and supporting extension of M can be embedded in $E(M)$ and because the property of being a distributive and supporting extension of M is invariant under M-isomorphism, it is enough to show that W has maximal elements.

W is partially ordered by inclusion and is non-empty. Consider any chain $\{L_i\}_i$ in W and let $L = \bigcup_i L_i$. Let $x \in M$ and $y \in L$, say, $y \in L_i$. Then $(M : y) + (y : x) = R$, since $M \subseteq L_i$ is distributive. Therefore $M \subseteq L$ is distributive. Furthermore if $\mathfrak{M}$ is a maximal ideal of R for which $L_{\mathfrak{M}} \neq 0$, then $(L_j)_{\mathfrak{M}} \neq 0$ for some j, and hence $M_{\mathfrak{M}} \neq 0$ since $M \subseteq L_j$ is supporting.

Therefore $M \subseteq L$ is supporting. Thus $L \in W$. Hence the result by Zorn's lemma.

COROLLARY 2. *An R-module M has maximal distributive extensions if, and only if, $M_{\mathfrak{M}} \neq 0$ for all maximal ideals $\mathfrak{M}$ of R.*

Proof. It remains to show that if $M_{\mathfrak{M}} = 0$ for some maximal ideal $\mathfrak{M}$ of R, then M has no maximal distributive extensions. But $M \subseteq N$ distributive implies $M_P \subseteq N_P = (N \oplus R/\mathfrak{M})_P$ distributive as R_P-modules, for all prime ideals P of R, $P \neq \mathfrak{M}$. Therefore if $M_{\mathfrak{M}} = 0$, $M \subseteq N$ distributive implies $M_P \subseteq (N \oplus R/\mathfrak{M})_P$ distributive for all prime ideals P of R, and hence $M \subseteq N \oplus R/\mathfrak{M}$ distributive by Lemma 2.1.

Let M be an R-module. It follows from Corollary 1 above that any two maximal distributive and supporting extensions of M are M-isomorphic if, and only if, M is a distributive and supporting submodule of an R-module $D(M)$ which contains an M-isomorphic copy of every distributive and supporting extension of M. In this situation, $D(M)$ is a unique (up to isomorphism) maximal distributive and supporting extension of M, and will be referred to as the "distributive hull" of M.

A ring is said to be *arithmetic* if the lattice of ideals of R is distributive. The local arithmetic rings are precisely the rings whose lattice of ideals is totally ordered; that is to say, the valuation rings. (This follows, for example, from the corollary to Proposition 1.1.) Thus a ring is local Noetherian and arithmetic if, and only if, it is a discrete valuation ring or a local Artinian principal ideal ring. More generally, as is well known, a ring is Noetherian and arithmetic if, and only if, it is a finite direct product of Dedekind domains and Artinian principal ideal rings. We show next that, over such a ring, every module has a distributive hull. We shall do this by first proving that, in the local case, either the injective hull or the module itself is a distributive hull. In fact this is true over all "almost maximal valuation rings," a wider class of local arithmetic rings. I am grateful to P. Vámos for suggesting this generalisation.

An "almost maximal valuation ring" can be defined as a local ring R for which every indecomposable injective R-module has a totally ordered lattice of submodules. Almost maximal valuation rings are valuation rings and they are precisely the class of local rings for which every finitely generated module is a direct sum of cyclic modules. (Details may be found in [3].) Thus every local Noetherian arithmetic ring is an almost maximal valuation ring.

LEMMA 2.3. *Let R be an almost maximal valuation ring. If M is a submodule of a cyclic R-module, then the injective hull of M has a totally ordered lattice of submodules.*

Proof. We may assume M is cyclic, so that $M \cong R/I$, where I is an ideal of R. But every ideal of R is irreducible, since the lattice of ideals of R is totally ordered. Therefore $E(R/I)$ is indecomposable [5, Proposition 2.28, Corollary 1]. Hence the lattice of submodules of $E(R/I)$ is totally ordered.

COROLLARY. *Let M be a module over an almost maximal valuation ring. Then either $E(M)$ or M is a distributive hull of M.*

Proof. Either M has no proper distributive extensions or, by the corollary to Proposition 1.1, M is a submodule of a cyclic module. In the latter case the submodules of $E(M)$ are totally ordered, and so $M \subseteq E(M)$ is distributive.

THEOREM 2.4. *Over a Noetherian arithmetic ring every module has a distributive hull.*

If M is a module over a local Noetherian arithmetic ring, the distributive hull of M is given by either M itself or the injective hull of M.

Proof. Consider first the local case. Let R be a local Noetherian arithmetic ring. Then R is an almost maximal valuation ring and so the result follows from the corollary to Lemma 2.3. However, we note for later reference that if M is a submodule of a cyclic R-module, then

(i) M is cyclic, because R is a principal ideal ring; thus $M \cong R/\mathfrak{M}^k$, where $\mathfrak{M}$ is the maximal ideal of R and $k \geqslant 0$; and

(ii) denoting the field of fractions of R by K in the case when R is a discrete valuation ring, the injective hull of M is

$$\begin{aligned} E(M) &= K, && \text{when } R \text{ is a discrete valuation ring and } k=0,\\ &= K/R, && \text{when } R \text{ is a discrete valuation ring and } k \geqslant 1,\\ &= R, && \text{when } R \text{ is a local Artinian principal ideal ring, not a field.} \end{aligned}$$

Thus we can see directly that the lattice of submodules of $E(M)$ is totally ordered, and hence that $E(M)$ is a distributive hull of M.

Consider now the general case. Let M be a module over a Noetherian arithmetic ring R and let $E(M)$ be an injective hull of M. Let W denote the set of submodules of $E(M)$ which are distributive and supporting extensions of M and let D be the submodule of $E(M)$ generated by the members of W. Suppose $\mathfrak{M}$ is a maximal ideal of R for which $M_{\mathfrak{M}} \neq D_{\mathfrak{M}}$. Then $M_{\mathfrak{M}} \neq L_{\mathfrak{M}}$ for some $L \in W$. Thus the $R_{\mathfrak{M}}$-module $M_{\mathfrak{M}}$ has a proper distributive and supporting extension and therefore, by the local case, the lattice of submodules of $E(M_{\mathfrak{M}})$ is totally ordered. But $E(M_{\mathfrak{M}}) = E(M)_{\mathfrak{M}}$ because R is Noetherian. Therefore $M_{\mathfrak{M}} \subseteq D_{\mathfrak{M}}$ is distributive and supporting. Thus $M_{\mathfrak{M}} \subseteq D_{\mathfrak{M}}$ is distributive and supporting for all maximal ideals $\mathfrak{M}$ of R.

Therefore $M \subseteq D$ is distributive and supporting, by Lemma 2.1. Hence by Proposition 2.2, D is a distributive hull of M.

3

Throughout this section R will be an integral domain and K will be the field of fractions of R. If M is a module over R, denote the torsion submodule of M by

$$T(M) = \{m \in M \mid rm = 0, \text{ some } r \in R, r \neq 0\}.$$

We recall that the rank $r(M)$ of M is the dimension of the K-vector space $K \otimes_R M$. Thus if m is an element of M, then $r(Rm) \leqslant 1$ and we have

$$r(Rm) = 0 \Leftrightarrow T(Rm) = Rm,$$
$$r(Rm) = 1 \Leftrightarrow T(Rm) = 0.$$

Note also that $r(M) = r(M_P)$ and $T(M)_P = T(M_P)$, for all prime ideals P of R.

PROPOSITION 3.1. *If $0 \neq M \subsetneqq N$ is a distributive extension of R-modules, then $r(M) = r(N) \leqslant 1$. Furthermore if this common rank is one, then $T(M) = T(N)$ and $\operatorname{Supp}(T(M)) \subseteq \{P \in \operatorname{Spec}(R) \mid M_P = N_P\}$.*

Proof. We first prove the proposition on the assumption that it is true over a local domain. Since $0 \neq M \subsetneqq N$, there are prime ideals P_1, P_2 of R such that $M_{P_1} \neq 0$ and $M_{P_2} \neq N_{P_2}$. By Lemma 2.1, $M_{P_i} \subseteq N_{P_i}$ is a distributive extension of R_{P_i}-modules for $i = 1, 2$. Therefore by the local case, $r(M) = r(M_{P_1}) = r(N_{P_1}) = r(N)$ and $r(M) = r(M_{P_2}) \leqslant 1$. Suppose the common rank of M and N is one. Then by the local case, $T(M_P) = T(N_P)$ for all prime ideals P of R, and if $T(M_P) \neq 0$ then $M_P = N_P$. Hence $T(M) = T(N)$ and $\operatorname{Supp}(T(M)) \subseteq \{P \in \operatorname{Spec}(R) \mid M_P = N_P\}$.

Consider now the case when R is a local integral domain. By the corollary to Proposition 1.1, we have $M \subseteq Ry$ for $y \in N - M$, and so

$$r(M) \leqslant r(Ry) \leqslant 1 \qquad \text{and} \qquad T(M) \subseteq T(Ry).$$

Therefore

$$r(Ry) = 1 \Rightarrow T(Ry) = 0 \Rightarrow T(M) = 0 \Rightarrow r(M) = 1,$$

and

$$r(Ry) = 0 \Rightarrow r(M) = 0.$$

Hence $r(M)=r(Ry)$ for all $y\in N-M$. Therefore

$$\begin{aligned} r(M)=0 &\Rightarrow M=T(M) \text{ and } r(Ry)=0, && \text{all } y\in N-M,\\ &\Rightarrow M=T(M) \text{ and } Ry=T(Ry), && \text{all } y\in N-M,\\ &\Rightarrow N=T(N)\\ &\Rightarrow r(N)=0. \end{aligned}$$

On the other hand,

$$\begin{aligned} r(M)\neq 0 &\Rightarrow 0\neq K\otimes_R M\subseteq K\otimes_R N \text{ is a distributive extension of}\\ &\qquad\qquad K\text{-modules}, \qquad \text{by Lemma 2.1},\\ &\Rightarrow K\otimes_R M=K\otimes_R N, \qquad \text{since } K \text{ is a field},\\ &\Rightarrow r(M)=r(N). \end{aligned}$$

It remains to show that $r(M)\neq 0$ implies $r(M)=1$ and $T(M)=T(N)=0$. Take an element $z\in N-M$. Then $r(M)=r(Rz)$ and $T(M)\subseteq T(Rz)$, so that

$$\begin{aligned} r(M)\neq 0 &\Rightarrow r(Rz)=1 \text{ and } T(Rz)=0\\ &\Rightarrow r(M)=1 \text{ and } T(M)=0\\ &\Rightarrow T(M)=0 \text{ and } r(Ry)=1, && \text{all } y\in N-M,\\ &\Rightarrow T(M)=0 \text{ and } T(Ry)=0, && \text{all } y\in N-M,\\ &\Rightarrow T(N)=0. \end{aligned}$$

COROLLARY 1. *An R-module of rank greater than one has neither proper distributive submodules nor proper distributive extensions.*

COROLLARY 2. *Let M be an R-module of rank one. If* $T(M)_{\mathfrak{M}}\neq 0$ *for all maximal ideals* $\mathfrak{M}$ *of R, then M has no proper distributive extensions.*

Over a Prüfer domain, the properties of distributive extensions of rank one modules given in Proposition 3.1 actually characterise such extensions.

PROPOSITION 3.2. *Let R be a Prüfer domain and let M be an R-module of rank one. A necessary and sufficient condition for* $M\subseteq N$ *to be distributive is that* $r(N)=1$, $T(N)=T(M)$ *and* $\mathrm{Supp}(T(M))\subseteq\{P\in\mathrm{Spec}(R)\mid M_P=N_P\}$.

Proof. By Proposition 3.1, we have only to prove the sufficiency.

Let P be a prime ideal of R not belonging to $\mathrm{Supp}(T(M))$. Then the R_P-modules $M_P\subseteq N_P$ are torsion-free of rank one and can therefore be regarded as submodules of K. But the R_P-submodules of K are totally ordered, because R_P is a valuation domain. Therefore by the corollary to Proposition 1.1, $M_P \subseteq N_P$ is distributive. Hence $M_P \subseteq N_P$ is distributive for all prime ideals P of R and so, by Lemma 2.1, $M \subseteq N$ is distributive.

4

Let M be a module over an integral domain R. Denote the set of zero-divisors of M by

$$Z(M) = \{r \in R \mid rm = 0, \text{ some } m \in M, m \neq 0\}.$$

Then $S_M = R - Z(M)$ is a multiplicatively closed subset of R and we may regard M as an R-submodule of the module of fractions $S_M^{-1}M$. Let X_M denote the set of non-zero prime ideals of R which are contained inside $Z(M)$.

LEMMA 4.1. *Suppose that M has rank one and that either $T(M) = 0$ or $X_M \subseteq \mathrm{Supp}(T(M))$. Then $S_M^{-1}M$ contains an isomorphic copy of every distributive extension of M. Furthermore if R is a Prüfer domain of Krull dimension one, then $S_M^{-1}M$ is a distributive hull of M.*

Proof. We first show that, as an $S_M^{-1}R$-module, $S_M^{-1}M$ has no proper distributive extensions. In the case $T(M) = 0$, this follows from the fact that $S_M^{-1}R$ is then a field. Suppose that $T(M) \neq 0$ and $X_M \subseteq \mathrm{Supp}(T(M))$. Then $S_M^{-1}R$ is not a field, and so every maximal ideal of $S_M^{-1}R$ is of the form $S_M^{-1}P$, where $P \in X_M$. But for such a P, $(S_M^{-1}M)_{S_M^{-1}P}$ is isomorphic with M_P and $T(M)_P \neq 0$. Hence $T(S_M^{-1}M)$ is locally non-zero at every maximal ideal of $S_M^{-1}R$. Therefore by Corollary 2 of Proposition 3.1, $S_M^{-1}M$ has no proper distributive extensions.

We can now deduce that $S_M^{-1}M$ contains an isomorphic copy of every distributive extension of M. If $M \subseteq N$ is distributive as R-modules, then $S_M^{-1}M \subseteq S_M^{-1}N$ is distributive as $S_M^{-1}R$-modules, and so $S_M^{-1}N = S_M^{-1}M$. But by Proposition 3.1, $T(M) = T(N)$, from which it follows that the canonical map $N \to S_M^{-1}N$ is an injection.

Now suppose that R is a Prüfer domain of Krull dimension one. It remains to show that $M \subseteq S_M^{-1}M$ is a distributive extension of R-modules. Let P be a prime ideal of R which meets S_M. Then from the fact that the Krull dimension of R is one, it follows that $S_M^{-1}R \otimes_R R_P \cong K$, the field of fractions of R. Thus $(S_M^{-1}M)_P \cong K \otimes_R M$, which is isomorphic with K since M has rank one. But the R_P-submodules of K are totally ordered. Therefore $M_P \subseteq (S_M^{-1}M)_P$ is distributive as R_P-modules. On the other hand, if P is a prime ideal of R which does not meet S_M, then $S_M^{-1}R \otimes_R R_P \cong R_P$, so that $M_P = (S_M^{-1}M)_P$. Hence $M \subseteq S_M^{-1}M$ is distributive, by Lemma 2.1.

PROPOSITION 4.2. *Let R be a Dedekind domain. Then the ideal class group of R is a torsion group if, and only if, $D(M) = S_M^{-1}M$ for all rank one R-modules M.*

Proof. The ideal class group of R is a torsion group if, and only if, whenever an ideal I of R is contained in the union of a family of prime ideals of R, I is actually contained in one of the prime ideals of the family [4, Theorem 2.2].

Suppose firstly that this is the case. Let M be a rank one R-module with $T(M) \neq 0$. Then $Z(M) = Z(T(M))$ and this is equal to the union of the associated primes of $T(M)$, since R is Noetherian. Therefore if $P \in X_M$, P is contained in an associated prime of $T(M)$, and so P itself is an associated prime of $T(M)$, since every non-zero prime ideal of R is maximal. It follows that P belongs to $\mathrm{Supp}(T(M))$. Hence $D(M) = S_M^{-1}M$ by Lemma 4.1.

Conversely suppose that the ideal class group of R is not a torsion group. Then since the complement of a union of prime ideals is multiplicatively closed, there is a prime ideal P of R and a family Y of prime ideals of R such that $P \subseteq \bigcup_{Q \in Y} Q$ and $P \notin Y$. Let M be the rank one R-module $R_P \amalg (\coprod_{Q \in Y} R/Q)$.

Let W be a prime ideal of R. If W does not meet S_M, then $M_W = (S_M^{-1}M)_W$. If W does meet S_M, then $W \neq P$ and $W \notin Y$, since $Z(M) = \bigcup_{Q \in Y} Q$, and so $M_W = K = (S_M^{-1}M)_W$, where K is the field of fractions of R. Therefore $S_M^{-1}M = M$.

However, $N = K \amalg (\coprod_{Q \in Y} R/Q)$ is a proper distributive extension of M, since $N_P = K$ is an R_P-module whose submodules are totally ordered, and $M_W = N_W$ for all prime ideals W of R, $W \neq P$. Thus $S_M^{-1}M$ is not a distributive hull of M.

The above results give, in particular, a description of the distributive hull of any abelian group of non-zero rank. To complete the picture we need to look at abelian groups of rank zero. The description of the distributive hull in this case is included in the next proposition. We recall the following well-known facts.

LEMMA 4.3. *Let R be a ring and let P, Q be prime ideals of R.*

(i) $R_P \otimes_R R_P \cong R_P$.

(ii) $R_P \otimes_R R_Q = 0$ *if, and only if, $P \cap Q$ contains no prime ideals of R. In particular, if R is Artinian, then $R_P \otimes_R R_Q = 0$ whenever P and Q are distinct prime ideals of R.*

(iii) *Suppose R is an integral domain with field of fractions K. Then $R_P \otimes_R R_Q \cong K$ if, and only if, $P \cap Q$ contains no non-zero prime ideals of R. In particular, if R is a Dedekind domain, then $R_P \otimes_R R_Q \cong K$ whenever P and Q are distinct prime ideals of R.*

PROPOSITION 4.4. *Let R and M be defined as follows: either* (i) *R is an Artinian arithmetic ring, not a field, and M is any R-module, or* (ii) *R is a Dedekind domain and M is an R-module of rank zero. Then the distributive hull of M is given by*

$$\left(\coprod_{P\in C} E(M_P)\right) \amalg \left(\coprod_{P\notin C} M_P\right),$$

where C is the set of maximal ideals P of R for which M_P is cyclic as an R_P-module.

Proof. Write $N = (\coprod_{P\in C} E(M_P)) \amalg (\coprod_{P\notin C} M_P)$. In each case M can be regarded as a submodule of $\coprod_P M_P$, where P runs over all the maximal ideals of R, and hence as a submodule of N. By Theorem 2.4 and its proof, $N = \coprod_P D(M_P)$.

Let Q be a maximal ideal of R. It follows from Lemma 4.3 and the description of the distributive hull in the local case, that $(D(M_Q))_Q = D(M_Q)$, and $(D(M_Q))_P = 0$ for all maximal ideals P of R, $P \neq Q$. Hence $N_Q = D(M_Q)$ for all maximal ideals Q of R. Therefore by Lemma 2.1, N is a distributive and supporting extension of M. But M has a distributive hull $D(M)$, by Theorem 2.4. Thus we may regard N as a submodule of $D(M)$. Then for each maximal ideal Q of R, $D(M)_Q$ is a distributive and supporting extension of M_Q with $N_Q = D(M_Q)$ lying in between, so that $N_Q = D(M)_Q$. Hence $N = D(M)$.

We consider now the situation over an arbitrary local Noetherian integrally closed domain. For any local ring R, with maximal ideal $\mathfrak{M}$, $\mathfrak{M} \subseteq R$ is always a distributive extension of R-modules. The proof of the next proposition shows that when R is a Noetherian integrally closed domain and the height of $\mathfrak{M}$ is greater than one, there are essentially no other distributive extensions of R-modules of non-zero rank. It would be interesting to know the corresponding situation for modules of rank zero.

Denoting the field of fractions of R by K, we have $\{x \in K \mid \mathfrak{M}x \subseteq R\} = R$ [7, Chap. V, Theorem 15]. Thus if X is an R-submodule of K, $y \in K$ and $k \geqslant 1$, then $\mathfrak{M}^k X \subseteq Ry$ implies $X \subseteq Ry$.

PROPOSITION 4.5. *Let R be a local Noetherian integrally closed domain of Krull dimension greater than one, and let $\mathfrak{M}$ be the maximal ideal of R. Let M be an R-module of non-zero rank. Then either $D(M) = M$, or $M \cong \mathfrak{M}$ and $D(M) \cong R$.*

Proof. If N is a proper distributive extension of M, then by Proposition 3.1, M and N are torsion-free of rank one and can therefore be regarded as R-submodules of the field of fractions of R. We claim that if y is any element of $N - M$, then $M = \mathfrak{M}y$ and $N = Ry$.

By the corollary to Proposition 1.1, $M \subseteq Ry$. Therefore M is finitely generated, because R is Noetherian. Let $x \in N$. Then $X = M + Rx + Ry$ is finitely generated and $M \subseteq X$ is distributive. Therefore $M = \mathfrak{M}^k X$, where $k \geqslant 1$ ([1, Proposition 9]). Thus $\mathfrak{M}^k X \subseteq Ry$. Hence by the above remark, $X \subseteq Ry$. Thus $x \in Ry$. Therefore $N \subseteq Ry$ and so $N = Ry$. Furthermore by Proposition 1.2, $M = \mathfrak{M}z$ for some $z \in N - M$. Hence the above argument gives $Rz = N = Ry$, and therefore $\mathfrak{M}z = \mathfrak{M}y$. Thus $M = \mathfrak{M}y \cong \mathfrak{M}$ and $N = Ry \cong R$.

5

We conclude with a remark about the definition "distributive hull." Supporting extensions were introduced in order to generalise to all modules results which otherwise would apply only to modules which are locally non-zero at every maximal ideal of the ring. In each of the results concerning distributive extensions of modules over a Noetherian ring, "supporting" can be replaced by the following property.

Let R be a ring (not necessarily Noetherian). Call an extension $M \subseteq N$ of R-modules *satisfactory* if given $y \in N$, $y \neq 0$, there exists $x \in M$ such that $\operatorname{Ann}(x) + \operatorname{Ann}(y) \neq R$.

Whereas "supporting" is too strong to be implied by "essential" (a counterexample is $\coprod_\infty \mathbb{Z}/2\mathbb{Z} \subseteq \prod_\infty \mathbb{Z}/2\mathbb{Z} = R$, as R-modules), it is easily seen that every essential extension is satisfactory. When R is Noetherian, a distributive extension of R-modules is supporting if, and only if, it is satisfactory. To see this observe that, for an extension $M \subseteq N$ of R-modules, the following implications are true:

(i) *Supporting implies satisfactory*;

(ii) *Distributive and satisfactory implies essential*;

(iii) *When R is Noetherian, essential implies supporting.*

Proofs. (i) Suppose $M \subseteq N$ is supporting. Let $y \in N$, $y \neq 0$. Then $\operatorname{Ann}(y) \subseteq \mathfrak{M}$ for some maximal ideal $\mathfrak{M}$ of R. Thus $N_{\mathfrak{M}} \neq 0$ and so $M_{\mathfrak{M}} \neq 0$. Hence there exists $x \in M$ such that $\operatorname{Ann}(x) \subseteq \mathfrak{M}$. Then $\operatorname{Ann}(x) + \operatorname{Ann}(y) \subseteq \mathfrak{M}$. Therefore $M \subseteq N$ is satisfactory.

(ii) Suppose $M \subseteq N$ is distributive and not essential. Then N has a non-zero submodule Y with $M \cap Y = 0$. For any $x \in M$ and $y \in Y$ we then have $(M : y) = \operatorname{Ann}(y)$ and $(y : x) = \operatorname{Ann}(x)$. Hence by Proposition 1.1, $\operatorname{Ann}(x) + \operatorname{Ann}(y) = R$. Therefore $M \subseteq N$ is not satisfactory.

(iii) If R is Noetherian and $M \subseteq N$ is an essential extension of R-modules, then $M_{\mathfrak{M}} \subseteq N_{\mathfrak{M}}$ is an essential extension of $R_{\mathfrak{M}}$-modules for all maximal ideals $\mathfrak{M}$ of R. Hence $M \subseteq N$ is supporting.

Thus in Theorem 2.4 and Proposition 4.4, $D(M)$ is a unique (up to isomorphism) maximal distributive and satisfactory extension of M.

In Propositions 4.2 and 4.5, $D(M)$ is in fact a unique (up to isomorphism) maximal distributive extension of M, because all extensions of modules of non-zero rank over an integral domain are supporting.

Finally observe that, whereas over commutative rings we found it more convenient to work with the notion of "supporting" than that of "satisfactory," the latter can be interpreted just as well over non-commutative rings. It follows from non-commutative versions of Proposition 1.1 and (ii) above, that over a ring which is not necessarily commutative, every module has maximal distributive and satisfactory extensions.

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